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THE FOURTH DIMENSION OF SPACE.

MR. SCHILLER'S summary of the discussion on this subject in the March number of this REVIEW indicates very clearly that the advocates of a fourth dimension latterly show a decided tendency to withdraw from some of their original claims, but it omits to notice a matter of very considerable importance in the problem which has received very scant attention on the part of the defenders of the doctrine, and has not been developed by its opponents, whose arguments often imply it. I allude to the purely logical principles at the basis of the matter. That these must first be satisfied, I think, is shown by several facts: (*a*) the tendency to abandon certain arguments in the case; (*b*) the absence of all deductive proof for a fourth dimension; (*c*) the want of data in experience to make the claim inductively rational; (*d*) the dependence upon analogies and symbolic conceptions as evidence.

But I shall waive all proof of the claim here made and allow the discussion itself to show its truth. The first step is to consider the general grounds upon which the doctrine is supposed to rest, as stated by some of its ablest advocates. They are: (*a*) the empirical nature of the Euclidean axioms; (*b*) the relativity of knowledge in general, shutting out a dogmatic denial of the hypothesis; (*c*) the Kantian doctrine of space, which, though it may prove the inconceivability (non-imaginable nature) of a fourth dimension, supports its possibility beyond the limits of experience; (*d*) the necessities of non-Euclidean geometry, especially for pseudo-spherical surfaces.

The first thing to be said regarding these arguments is that, if the laws of logic have first been respected, they may be entitled to some weight, but if these laws have been violated, the arguments can count for nothing. Hence I wish to call attention to certain irrelevancies in them, in order to show how the prior conditions of all intelligible discussion in this problem are certain logical principles that reveal very clearly where the confusion originates in the controversy. This irrelevancy is that which connects the question with the problems about empiricism, intuitionism, transcendentalism, realism, idealism, etc. These, in fact, have nothing to do with the matter until after we know the logical terms of the problem. In all cases we have to do with certain conceptions which carry with them the same implications *logically*, whether we choose to regard them as real or ideal, objective or subjective, empirical or intuitive. What I have to consider, therefore, is the logical use made of the conceptions 'space,' 'property,' 'dimension,' 'mathematics,' etc., in the attempt to prove a fourth dimension.

Now I shall first state a few simple logical principles upon which I shall proceed, and which determine the limits of legitimate reasoning in this problem. They are perfectly familiar laws to the logician, but seem to be wholly ignored by mathematicians. They are summarized in this one proposition: *The transfer of predicates and implications from one conception to another is limited to a qualitative identity between them.* This can be clearly illustrated by reference to the relation between certain conceptions and certain tendencies in the growth of knowledge.

Concepts express certain definite relations between genus and species, and between different species. We may express this generally by the formula that their extension varies inversely with their intension. In common parlance, this is only to say that the number of individuals denoted by the genus is greater than the number denoted by the species, while the number of qualities denoted by the species is greater than that denoted by the genus. It is not necessary here to assert or defend the *absolute* universality of this rule, but only that it is unquestionable in a certain class of conceptions, and these are the conceptions with which we have to deal in our present discussion. Now the plain simple rule here is that we can never transfer the differential predicates of the species to the genus, and also that general formulas have to be modified to suit the differentia of the species. For example, I cannot transfer the differential quality expressed by 'Caucasian' over to the concept 'man,' and I cannot express the meaning of 'Caucasian' by stopping with the predicates of the term 'man.' These are simple truisms, but they get great importance in connection with discussions that violate them, owing to the additions made to knowledge by intellectual progress.

The development of knowledge involves two different changes in conceptions. They may be widened or they may be narrowed in their import. These two processes are known to the logician as generalization and specialization. Until the new meaning becomes the only and fixed import of the term, it gives rise to equivocation. In this way an interchange of predicates and implications will occur, and often unconsciously. But this is the illusion for which intelligent men are required to be on the alert. This difficulty, however, is greatly increased by the several ways in which concepts may grow in denotation and meaning. First, concepts may increase or decrease in nothing but quantitative import. Secondly, they may increase or decrease only in qualitative import. Thirdly, quantitative and qualitative import may vary in an inverse ratio with each other. Thus the first of these processes occurs when a new individual or species is added to the genus, or an old one withdrawn, without affecting the conferentia (common qualities) expressed by it. Here the change does not affect the transfer of predicates. It is purely quantitative, and this is the peculiarity of all purely mathematical concepts. In the second process the change occurs when a new quality is added, or an old one withdrawn from a concept, without changing its quantitative import or extension. This change also does not affect the truth or universality of old propositions, and a transfer of predicates will not take place. No equivocation, however, will occur. But it is the third form that causes all the trouble. In this the extension may increase at the expense of the intension and *vice versa.* This occurs when a new species is added to a genus so as to decrease the intension, or a species withdrawn so as to increase the intension. In such cases the transfer of predicates cannot take place. Or, to summarize the discussion, when conceptions change quantitatively, but not qualitatively, the transfer of predicates can be made with perfect logical impunity. When they change qualitatively, but not quantitatively, new predicates are added which are differentially distinct from the old ones, but there is no occasion for a transfer. But when quantitative and qualitative import vary inversely, a transfer of predicates cannot be assumed without proof. Now, since mathematics is limited to the quantitative concepts or qualities, and logic extends to both quantitative and qualitative meanings of terms, it is apparent how they come into relation with each other, and how a habit contracted in the quantitative determinations of mathematics may pass over to cases where the changes are qualitative as well. In mathematics we either do not deal at all with genus and species, but with whole and part, which are qualitatively identical; or, if we call the broader and narrower concepts 'genus' and 'species,' they are still qualitatively of the same import. But in logic, besides whole and part we deal with genus and species, which are qualitatively different from each other. The consequences of this may be brought out by illustration.

The instance is taken from the fluctuations in the conception 'metal.' In physics and chemistry brass and bronze are not metals; in common parlance they are. Now in scientific usage I can say, "All metals are elements"; in common parlance I cannot say it, because brass and bronze are compounds. Here, with the extension of the term 'metal,' I cannot carry the predicate of its narrower import with me. With this increase of extension, 'element' becomes the differentia of a species. Hence in any case where we undertook to define the differential quality of brass and bronze, we should have to call it non-elemental, not having any right to use the term 'element' to describe it, unless it also be generalized. On the other hand, the same process is illustrated by another interesting generalization of the same term. At one time it was assumed that a specific gravity greater than water was an essential property of metals. It was conceived as essential to a metal that it sink in water.

This conception excluded at least three of the alkali metals, potassium, sodium, and lithium. But the discovery that these substances possessed metallic lustre and probably other metallic properties, resulted in extending the class 'metals' to include them while diminishing the conferentia, and this in spite of the fact that their specific gravity is *less* than water. Now we have here a generalization of the term 'metal' in which we cannot carry with us the old proposition, "All metals sink in water." This relation now becomes the differentia of a species, and is no longer a conferentia. If the reverse process had taken place, it would have been necessary to have added a new predicate to the species.

The value of these principles will be apparent in the examination of the argument for a fourth dimension, most especially as it appears in Helmholtz' celebrated articles in *Mind*,[1] which have done more than anything else to make philosophers take the subject seriously. The first illusion of which he and mathematicians generally have been the victims, is not one which comes under the principles just enunciated, but is nevertheless an important weakness in their argument. It is the transference to the conception of space of assumptions and conceptions that are true of material substance. Now the mathematician tells us that geometry deals with the properties of space. Dimension is said to be one of these properties, if not the only one, and as there are admittedly three of these dimensions, the limitations of our empirical knowledge at once suggest the possibility of more of them. The only problem is to produce the facts which will either prove their real existence, or show that they are thinkable and possible. The fact that we know of no limits to the properties of matter, and that discovery constantly shows additions to our knowledge of new properties, forces, or modes of action (the Röntgen rays, for example), or at least new phenomena, stands in good stead to shut off dogmatic denials of other than the known dimensions of space. But it is precisely here that the illusion occurs. The mathematician permits himself to be fooled by words, and pays no attention to their real import. He assumes without criticism that the relation between space and its dimensions is the same as that between matter or a metaphysical substance and its properties. This assumption may be absolutely denied, and I certainly deny the right to make it. The illusion arises first from the language about the 'properties' of space, and secondly from identifying 'properties' with dimensions, while distinguishing tacitly between space and its 'properties' on the one hand, and space and its dimensions on the other. Metaphysical realities, subjects or substances, like matter, spirit, ether, etc., may have any number of properties, known and unknown. But we have no *a priori* right to carry this possibility over to space, because no one entertains for a moment the supposition that it is a metaphysical substance like matter or other reality. It is qualitatively distinguished from such conceptions. It may be that space possesses an indefinite number of properties, but we can neither assume the fact or possibility from what we hold to be true of matter, mind, and other subjects or substances, nor assume that we can treat the conception of space in the same way. We have to prove on other grounds that the conception of space is subject to the same treatment. What I contend for is, that we cannot logically pass, as the mathematicians do, from one of these conceptions to the other, and that propositions in the two cases, notwithstanding their formal resemblances, do not have the same meaning and implication unless proved on other grounds than this formal identity ; so that the very first step in the argument for a fourth dimension is vitiated by presumptions which have no right to exist.

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[1] Vol. I, p. 301; vol. III, p. 212.

The whole problem of the advocates of a fourth dimension is to find a basis for non-Euclidean geometry. Euclidean geometry is admittedly based upon the three dimensions, and they assume that this new kind of geometry requires a new differential principle. They are at least formally correct, according to the principles established regarding the relation between genus and species or between different species. But we must examine what difference they assume to exist between the two kinds of geometry. If the two are the same, the demand for a fourth dimension would be absurd, according to their own admission. If they are different, if non-Euclidean geometry is different from Euclidean, the difference must be either quantitative, or qualitative, or both. If it be merely quantitative, the qualitative principle or condition is the same as the Euclidean; if it be qualitatively different, then the new principle must be a new quality, a new property of space, as the fourth dimension is supposed to be. If the difference be both quantitative and qualitative, then the distinction between Euclidean and non-Euclidean geometry is not absolute, but they interpenetrate in the dimensions determining Euclidean geometry. After ascertaining the alternatives between which we are placed, the only question that remains to determine concerns the conceptions of the problem entertained by non-Euclidean mathematicians. The second alternative is the one maintained; and this with its qualitative distinction between the two kinds of geometry, implies that the fourth dimension must be a new quality or property of space, or qualitatively different from the other dimensions. The first alternative is fatal because it limits the difference to quantity, the qualitative principle remaining the same, so that but one rational course is open to the mathematician, which is to affirm a difference of kind. We start, then, with the assumption that non-Euclidean geometry requires a principle for its basis qualitatively distinct from that of Euclidean geometry. What is the consequence of this step?

The basis of geometry is said to be the 'properties of space.' We may ask what is meant by the 'properties' of space, and this question proposes the problem of determining whether 'space' is synonymous with its 'dimensions,' or may include other 'properties' than dimension, and whether its 'properties' are the same as its dimensions. This problem ought first to be solved by the non-Euclidean geometer before he takes any other step. But I know of no attempt to do this. He has two alternatives. He may limit the intension of space to the dimensions, or he may extend it to include other properties than dimension, such as penetrability and divisibility or indivisibility. (I hold that space is absolutely indivisible, though it is usually spoken of as divisible. In reality it is body that is divisible.) Now if space denote or imply other properties than dimension, we may ask what evidence is there that the so-called 'fourth dimension' is a dimension at all. The non-Euclideans agree that their geometry is based upon the 'properties of space.' This limits them to two alternative conceptions, assuming that the two geometries must be distinguished. Either 'space' denotes other properties than dimension, or in being limited to dimension we must suppose, as they do, that the fourth dimension is qualitatively different from the other three. The supposition that the 'fourth dimension' is different in kind from the other three, and at the same time that space denotes only the three dimensions, would imply that non-Euclidean geometry is non-spatial; that is, not based upon space at all, which is contrary to the original assumption. But, taking the two conceptions just mentioned, it should be noticed that the first may justify us in selecting some other property than dimension for the basis of non-Euclidean geometry. What reason have the non-Euclideans for distinguishing between the fourth dimension and some other property not a dimension at all, especially as they admit that this new 'dimension' cannot be pictured or represented in experience? Taking the second alternative, we find that a generalization either of the term 'space' or of the term 'dimension' has been made. If of the term 'space,' the 'fourth dimension' either becomes a non-dimensional property, or the basis of geometry has been altered in its conception, which might enable us to take any quality of anything as the principle of non-Euclidean geometry.

Let me make the case clearer by another form of statement. If we assume the qualitative difference between Euclidean and non-Euclidean geometry, there are four conceptions of space to be considered, three of them absolutely necessary to satisfy this assumption: (1) Space = three dimensions; (2) space = three plus the fourth dimension or n dimensions; (3) space = three dimensions plus other properties; (4) space = four or n dimensions plus other properties.

Taking space in the first of these three conceptions, the fourth dimension must make non-Euclidean geometry non-spatial, which is contrary to the supposition. On the third conception, the principle of non-Euclidean geometry is not a dimension, but some other property. Assuming the fourth conception, the non-Euclidean geometer must show the distinction to be made between the fourth dimension and other properties, especially that this dimension is qualitatively different from the other three. If not qualitatively different, non-Euclidean geometry falls to the ground as anything more than a modification of Euclidean geometry. This leaves, as the only alternative for the non-Euclidean, the second, which is the conception, and the only conception, of space that can present even a plausible claim in favor of a fourth dimension for the principle of non-Euclidean geometry.

Now, in regard to this second conception of space, the first remark is that it is an extension of the meaning involved in the first. But passing this by as unimportant, though necessary to non-Euclidean geometry, the second remark is that the term 'dimension' is either generalized in its import qualitatively, or it is a name to denote a non-dimensional property. The only other alternative is to hold that the three dimensions and the fourth are not different from each other. I want, therefore, to show the logical consequences to the doctrine from each one of these alternatives.

The assumption is that the fourth dimension is qualitatively different from the other three dimensions. It is, therefore, a species in contradistinction to them as other species. Now, when the term 'dimension' includes all of them, it denotes a common property, the conferentia, or genus; and cannot be used to denote the species. This would be in violation of the principle of logical division, which is that the same conception cannot denominate both genus and species. Assuming that it denotes only the genus, or common quality of all the dimensions, we find that both Euclidean and non-Euclidean geometry are based upon the same quality of space, which is contrary to the supposition. On the other hand, if it denote only a species, it must be limited either to the three dimensions or to the fourth, if a qualitative distinction between them is to be maintained. If limited to the three, then it is not legitimate to call the 'fourth dimension' a dimension at all, and non-Euclidean geometry would be based upon a non-dimensional property, say penetrability or indivisibility, which is contrary to the original supposition. If it be limited to the fourth, then the other three are not 'dimensions' properly considered, and Euclidean geometry would be non-dimensional, which is also contrary to the supposition. The only alternative left is to apply the term equally to all four dimensions. But this identifies them qualitatively and breaks down the distinction between Euclidean and non-Euclidean geometry, which again is contrary to the supposition, unless we go outside of space altogether for the basis of the latter, which again contradicts the first assumption. Such a fatal set of dilemmas could hardly have been suspected on a first glance at the controversy; but they are there as long as we use the word 'dimension' in the case, and distinguish qualitatively between Euclidean and non-Euclidean geometry.

The fundamental fault of the mathematicians has been in extending the meaning of the term 'dimension' by adding a new species and calling it by the same name as the old. This mistake never occurs in the natural sciences. When a new species is discovered, increasing the extension of the genus, a new name must be adopted expressing the differentia by which this species is distinguished from the others. If the fourth dimension be a new species qualitatively different from the others, it should either not be called a dimension at all, or something should be indicated to determine the differentia by which it is presumably differentiated from the others. We may generalize the term 'dimension' if we choose, but we must not carry with it the differentia which separates the species; and we are equally forbidden to employ the same term for the species. The reply to this criticism would be that the differentia is expressed in the number of the dimension, and this reply is formally legitimate. But it is fatal in two respects to the hypothesis of a new dimension qualitatively determined. First, if number be the differentia of the species, it is purely quantitative, and the basis of non-Euclidean geometry is not qualitatively distinguished from the Euclidean. Secondly, if the conception 'fourth,' *i.e.*, number, determines a qualitative differentia, then the first, second, and third dimensions should be qualitatively different from each other, which is contrary to the supposition of Euclidean geometry. They are assumed to express the same commensurable quality, while their supposed differences are only relations of direction from a given point.

The language easily lends itself to an illusion, because it is formally the same as that in which qualitative differences are actually expressed or implied. But in mathematics our first duty is to remember that our conceptions are primarily quantitative, and that when we go beyond purely quantitative distinctions we are transcending mathematics altogether.

What I have said here about the illusory nature of the language in the case is beautifully illustrated in the expression, "Space has dimension." This proposition resembles the ordinary intensive judgment (such as "Man is wise," where it is possible to have other predicates in the same subject) only when we conceive the subject, space, as possibly having other properties than dimension; but when the term 'space' is made convertible with 'dimension,' as is usually or always the case in mathematics, we should either not assume that "Space *has* dimension," or when using the phrase we should recognize logically its true import, namely, that "Space *is* dimension." For geometry, space and dimension are the same, and hence in reality to assert the existence of a fourth dimension is equivalent to saying that the three dimensions have a fourth or *n* dimension, or that the three dimensions are four or *n* dimensions. The absurdity of this is apparent, but it is concealed by the formal correctness of the proposition, "Space has properties," or, "Space has dimension." But the moment we see that, for geometry, space and its dimensions are the same, we are forced to recognize that the fourth dimension becomes a predicate of the other three dimensions, which is contrary to the supposition of non-Euclidean geometry.

We are now prepared to examine some concrete fallacies and illusions of the same kind committed by Helmholtz in the celebrated articles in *Mind* already referred to, on the "Origin and Meaning of Geometrical Axioms." His argument here is to prove the empirical nature of geometrical axioms, and thus to avail himself of the inference, which the limitations of empiricism justify, that there are possibly other data in existence than the three known dimensions. In order to establish this empiricism, he undertakes to show that the axioms do not have the universal and necessary application which they are supposed to have. In this procedure he is half conscious of the principle that I have here laid down about the impossibility of transferring differential predicates when an increase in the extension of our concepts takes place, and the force of his argument derives all its influence from the truth of this principle. But he immediately violates the principle by equivocations which are due to specializing terms without reckoning with the logical consequences of the act. Let us examine his procedure briefly.

He calls attention to the assumed universality of the axiom about a straight line being the shortest path between two points, only to show that it is not true to a being living on a curved surface, to whom a *curved* line is the shortest distance between two points. This fact is supposed to set aside the universality of the Euclidean axiom. But there is a curious illusion in this claim which can be dispelled in two ways. In the first place, there is an equivocation in the word 'shortest.' Mathematically speaking, the Euclidean axiom still remains true to any being living on a spherical surface, though it may not be *physically* true. Even if it be assumed that such a being could not move directly at all from one point to the other, the distance physically and temporally the shortest to him would be a curved line, but this truth has nothing in it to contradict or modify the Euclidean axiom which still remains true mathematically where we have to do with *pure* space relations and not with qualities other than the spatial. Secondly, if the being living on the sphere *knew* that this surface was curved, it would recognize the Euclidean axiom, and, if influenced by any economic motives prevalent about walking on the diagonals of street corners, would sigh for the *physical* capacity to conform to mathematical principles. But if it did not know that the surface was a curved one, *it could not draw any distinction between a straight and a curved line.* Its mathematical and physical conceptions of 'shortest' would coincide, so that *straight* and *curved* would mean the same thing, and the Euclidean axiom would still remain. But Helmholtz happens to know the difference between mathematical space and physical body, and by an equivocation in the use of 'shortest' can obtain an apparent limitation to this axiom, when applying it from the standpoint of his own assumed knowledge compared with that of a being supposed to be ignorant of his point of view. But the equivocation does not help the matter, and the ignorance of the other being does not interfere with the truth of the Euclidean axiom.

A long examination of another instance by Helmholtz, impeaching the universality of the proposition that the sum of the angles of a triangle is equal to two right angles, might be given, but it is sufficient to take note of two omissions in order to vitiate the conclusion that he wishes to draw from his result. In the first place, he confuses two different degrees of extension in the use of the term 'triangle,' one limited to plane and the other including spherical triangles, which shows only that the universality of a proposition is never intended to extend beyond its subject. The proposition about the sum of the angles remains forever true within these limits, and Helmholtz forgets that the language, while it may include spherical triangles, is *conceived* by the mathematician concretely to mean plane triangles. He can also obtain a universal proposition for both. Secondly, Helmholtz fails to see that, although a modification of the formula or principle in this proposition is required to meet the conditions of a new species, this modification is purely *quantitative,* not qualitative, and hence the analogy lends no support to the qualitative difference implied or asserted in the fourth dimension as the basis of the relations in pseudo-spherical surfaces. There is an illusion also in assuming or insinuating that pseudo-spherical surfaces are more than quantitatively different from plane and spherical surfaces, so far as commensurable quality is concerned.

The effect of the equivocation in the use of the word 'dimension' is apparent in another way, to which attention must be called. If there is anything upon which mathematicians and mankind generally are agreed, it is that space has at least three dimensions, Euclidean geometers and most others holding that it has *only* three dimensions. But I think both can be denied, without favoring the contention of non-Euclidean mathematicians that there is a fourth dimension in any sense in which they are understood to affirm it. In denying the existence of three dimensions, we have two alternative affirmative propositions, both of which may be true if we assume two meanings for the term 'dimension.' They are: (1) that space has only *one* dimension; (2) that it has an indefinite or infinite number of dimensions. This claim is borne out by the fact that, when we speak of space as having 'dimension,' we express a single quality which is divided up into 'three dimensions,' without implying that the species are qualitatively different from their base, but are only relations of the same quality to different points of view. In fact the 'three dimensions' are properly defined and reducible to *commensurable quality* in which the units are always the same in each dimension. The three dimensions, therefore, cannot qualitatively differ from this without losing their commensurable nature. Why, then, are they called 'dimensions,' as if they were species of a genus? The answer to this question must be, either that the term is illegitimate altogether, or that it expresses only certain quantitative relations having mathematical convenience in the mensuration of bodies. Both alternatives are fatal to the supposition of a fourth 'dimension' in a qualitative sense without either going outside the meaning of dimension as denoting commensurable quality, or going outside the conception of space, which are both contrary to the supposition of non-Euclideans.

The supposition that there are three dimensions instead of one, or that there are only three dimensions, is purely arbitrary, though convenient for certain practical purposes. Here the supposition expresses only differences of relation ; that is, *differences of direction from an assumed point.* Thus, what would be said to lie in a plane in one relation, would lie in the third dimension in another. There is no way to determine absolutely what is the first, second, or third dimension. If the plane horizontal to the sensorium be called plane dimension, the plane vertical to it will be called solid, or the third dimension, but a change of position will change the names of these dimensions without involving the slightest qualitative change or difference in meaning. Moreover, we usually select three lines or planes terminating vertically at the same point, the lines connecting the three surfaces of a cube with the same point, as the representatives of what is meant by three dimensions, and reduce all other lines and planes to these. But interesting facts are observable here. (1) If the vertical relation between two lines be necessary for defining a 'dimension,' then all other lines than the specified ones are either not in any dimension at all, or they are outside the three given dimensions. This is denied by all parties, which only shows that a vertical relation to other lines is not necessary to the determination of a dimension. (2) If lines outside the three vertically intersecting lines still lie in dimension, or are reducible to the other dimensions, they may lie in more than one dimension at the same time, which after all is a fact. This only shows that qualitatively all three dimensions are the same, and that any line outside of another can only represent a dimension in the sense of *direction* from a given point or line, and we are entitled to assume as many dimensions as we please, all within the 'three dimensions.'

This mode of treatment shows the source of the illusion about the 'fourth dimension.' The term in its generic import denotes commensurable quality and denotes only one such quality, so that the property supposed to determine non-Euclidean geometry must be qualitatively different from this, if its figures involve the necessary qualitative differentiation from Euclidean mathematics. But this would shut out the idea of 'dimension' as its basis, which is contrary to the supposition. On the other hand, the term has a specific meaning, which, as different qualitatively from the generic, excludes the right to use the generic term to describe them differentially, but if used only quantitatively, that is, to express *direction*, as it in fact does in these cases, involves the admission of the *actual*, not a supposititious, existence of the fourth dimension, which again is contrary to the supposition of non-Euclidean geometry. Stated briefly, dimension as commensurable quality makes the existence of a fourth dimension a transcendental problem, but as mere direction an empirical problem, and the last conception satisfies all the requirements of the case, because it conforms to the purely quantitative differences which exist between Euclidean and non-Euclidean geometry, as the very language about 'surfaces,' 'triangles,' etc., in spite of the prefix 'pseudo,' necessarily implies. If the difference be made qualitative, neither the conception of direction will satisfy the case, because this is quantitative, nor that of dimension, because the fourth dimension would have to be *non*-dimensional. The simple illusion of Helmholtz lies in the confusion of dimension, now denoting commensurable quality, with direction, now denoting certain quantitative relations, and he merely carries this confusion over to the 'fourth dimension,' with the implications of transcendentalism in its qualitative differentiation from the others.

Why Helmholtz should have been guilty of this confusion it is hard to say, when we remember his own conception of the basis of geometry. In the very article above referred to, he says: "In conclusion, I would again urge that the axioms of geometry are not propositions pertaining only to the pure doctrine of space. As I said before, they are concerned with quantity." If geometry can be based upon the notion of quantity as well as space quality, he ought to have seen at once that his 'fourth dimension' did not require to be a new quality, but only a new quantitative relation of the one quality of space, which it in reality is. Distinguish between 'dimension' as commensurable quality and the use of the term to denote directional relations, and the problem is solved. The fourth and even *n* 'dimensions' can be admitted as empirical *facts*, and there will be no necessity for showing the empirical nature of geometrical axioms, in order to obtain an *a priori* presumption, from the limitations and indefinite capacities of experience, in favor of a possible existence for transcendental properties of space.

There is one more illusion growing out of this confusion of 'dimension' with direction. It relates to the movements of points, lines, and figures, assumed by mathematicians in representing the various relations expressed by Euclidean space. The motion of a point is said to produce a line in one dimension; the motion of a line about one end produces a plane, and the motion of a plane about one of its sides will produce a solid, or the third dimension. The 'fourth dimension' is demanded for a certain motion of a solid! But we may say first that, in mathematical parlance, a point cannot be made to move, nor can a line or a plane. Only bodies can move. This may be admitted to be quibbling, but it calls attention to the fact that, if mechanical motion is to determine the matter of dimension, the motion of a 'point,' or 'atom,' must be in more than one 'dimension' at a time. A solid, being in three dimensions, will move in them, and, if it gets out of them, will either not be a solid at all, or, if it is in the 'fourth dimension,' we should require a transcendental physics as the basis of non-Euclidean geometry, and this is not in the contract of the mathematician, but only a new property of space. But to dismiss quibbling, if we accept the fact that the dimensions can be constructively represented as described, why assume that a point can move only in one dimension, a line in two, and a plane in the third? From what has been said about the relative and interchangeable nature of the dimensions, any one being the other according to point of view, and from the fact that the motion of a point must pass *through* what is called the third dimension and also exists in a plane at the same time, it is evident that even a moving point must imply all three dimensions. It cannot move in all three *directions* at the same time, but the whole commensurable quality of space is implied by the existence of a point, a line, and a plane, as well as a solid. Hence geometry, constructive and symbolic, is based, not upon dimensions as commensurable quality, but upon dimensions as directions, and in this way creates no presumptions in favor of any new commensurable quality. To argue for it is simply one of those equivocations which ought not to deceive a common schoolboy, not to say anything of men with the reputation of Helmholtz and Riemann.

Several other similar illusions might be pointed out, such as Helmholtz' language about *flat space* and *curved space*, but I shall not discuss them here. They are either a confusion of the abstract with the concrete, or of quantitative with qualitative logic ; and after our lengthy exposure of this latter all-pervading fallacy, it is not necessary to do more than to reiterate the one important rule that qualitative differences can never be expressed by the same term, so that all this discussion about a fourth dimension is simply an extended mass of equivocations turning upon the various meanings of the term 'dimension.' This, when once discovered, either makes the controversy ridiculous or the claim for non-Euclidean properties a mere truism, but effectually explodes the logical claim for a new dimensional quality for space, as a piece of mere jugglery in which the juggler is as badly deceived as his spectators. It simply forces mathematics to transcend its own functions as defined and limited by its own advocates, and to assume the prerogatives of metaphysics. With the non-Euclideans it would become a science of quality as well as, or instead of, quantity, and would hardly stop with Helmholtz' empiricism for an argument in favor of its transcendental 'dimension.'

I have intended this exhaustive logical criticism as a precaution against a great deal of crazy metaphysics which might support itself upon the authority of men like Helmholtz and Riemann. Occultism simply revels in the doctrine of a fourth dimension, and is absolved from the duty of proving it *in se* by the authority of presumably sane scientific men ; and while it may be sufficient simply to laugh at the pretensions of the occultist, and while it only dignifies his speculations seriously to consider them, there are some at least quasi-genuine phenomena which throw the reins to madhouse theories, when both parties soberly discuss the claims for a fourth dimension and remain wholly ignorant of the logical principles, which not only vitiate the argument for the existence, or even possibility, of this 'dimension,' but make the talk about it mere child's play. In taking this position, however, it is not necessary to deny the fact of other than the known properties of existence, nor to deny that there is more than is dreamt of in any of our philosophies, but only that the logical terms of the problem take us wholly beyond the limits of geometry and mathematics for our 'meta-dimension.' Not only must we distort and change our conception of space, but we require equally to modify that of geometry and mathematics, so that they cease to deal with mere quantity and are made to share the precarious fortunes of metaphysics. We may take this course if we like, but our science would lose its much boasted certitude by the change, and would very soon turn into a fool's paradise. We cannot limit mathematics by definition to the consideration of pure quantity, and then introduce into our data qualitative differentials which bear no quantitative import but the name. If we do this, the futility of our procedure is only concealed by one of the simplest of illusions, unless it is our distinct purpose to base mathematics upon a system of metaphysics which is as fanciful as wonderland. An equivocation is a poor compass, when we set out on Kant's shoreless ocean in search of a harbor, and, if we discover its character before we make the venture, we shall be all the wiser for it. But without equivocation we can in no case accomplish any more than the man in Mother Goose, who "ran fourteen miles in fifteen days and never looked behind him," only to find in the end that he was just where he had started.

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